

CS 331, Fall 2026

Lecture 14 (10/14)

Today: - Cuts

- Maxflow
- mincut
theorem

- Flow algos

Cuts (Part V, Section 4.1)

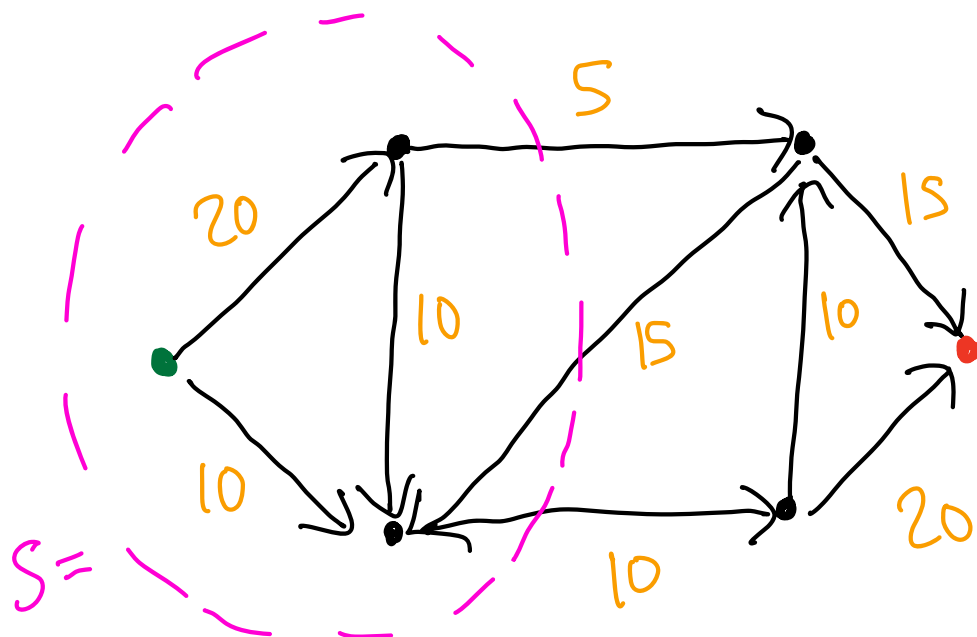
"Dual" problem to flows.

For $S \subseteq V$:

$$\text{cut}(S) = \sum_{\substack{(u,v) \in E \\ u \in S \\ v \notin S}} C(u,v)$$

"total capacity crossing from S to $V \setminus S$ "

Example



We only track
from $S \rightarrow V \setminus S$

$$\text{cut}(S) = 10 + 5 = 15$$

$$\text{cut}(V \setminus S) = 15$$

Say that S is an S - T cut if

$$s \in S, \quad t \notin S$$



S separates s from t

Historical context

Flows / cuts first formulated in classified report (1954) modeling Soviet network

Harris-Ross were interested in ^{"cut"} disrupting ...

(Declassified by Schlesinger, '05)



Maxflow - mincut theorems (Part V, Section 4.2)

max	$f(s)$
$0 \leq f_e \leq c_e$	(feasible)
$f(v) = 0$	
$\forall v \notin \{s, t\}$	$(s \rightarrow t \text{ flow})$

$s \rightarrow t$ maxflow

min	$cut(S)$
$S \subseteq V$	
$s \in S$	
$t \notin S$	$(s \rightarrow t \text{ cut})$

$s \rightarrow t$ mincut

Claim: for any $G = (V, E, \underbrace{c}_{\text{positive}})$, $s \neq t \in V$

$$S \rightarrow \text{maxflow} = S \rightarrow \text{mincut}$$

(Strong maxflow-mincut theorem)

Sanity Check: if no $S \rightarrow T$ path, both = 0

Weak maxflow-minicut theorem:

$$S \rightarrow T \text{ maxflow} \leq S \rightarrow T \text{ mincut}$$

Proof: Flow must get from S to $V \setminus S$

It only has $\text{cut}(S)$ to do so.

More formally, $\partial f(S) = \sum_{u \in S} \partial f(u)$

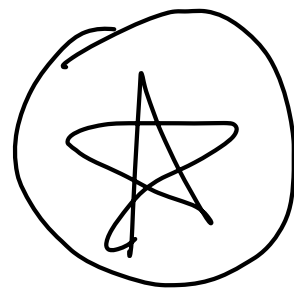
$$= \sum_{u \in S} \sum_{(u,v) \in E} f_{(u,v)} - \sum_{u \in S} \sum_{(v,u) \in E} f_{(v,u)}$$

$$= \sum_{\substack{(u,v) \in E \\ u \in S \\ v \notin S}} f_{(u,v)} - \sum_{\substack{(v,u) \in E \\ u \in S \\ v \notin S}} f_{(v,u)}$$

$$\leq \sum_{\substack{(u,v) \in E \\ u \in S \\ v \notin S}} c_{(u,v)} = \text{cut}(S)$$

For strong maxflow-mincut:

need flow f , cut S s.t.



$$1) f_{(u,v)} = c_{(u,v)} \quad \forall (u,v) \in E \quad \begin{matrix} u \in S \\ v \notin S \end{matrix}$$

$$2) f_{(v,u)} = 0 \quad \forall (v,u) \in E \quad \begin{matrix} u \in S \\ v \notin S \end{matrix}$$

Residual graph

"What are all flows
I can add s.t. stay feasible?"

Let $(u,v) \in E$.

- If $0 < f_{(u,v)} < C_{(u,v)}$:

Add (u,v) to G^f , capacity = $C_{(u,v)} - f_{(u,v)}$

(v,u) to G^f , capacity = $f_{(u,v)}$

- If $f_{(u,v)} = C_{(u,v)}$:

Only add backward edge

(v,u) to G^f , capacity = $f_{(u,v)}$

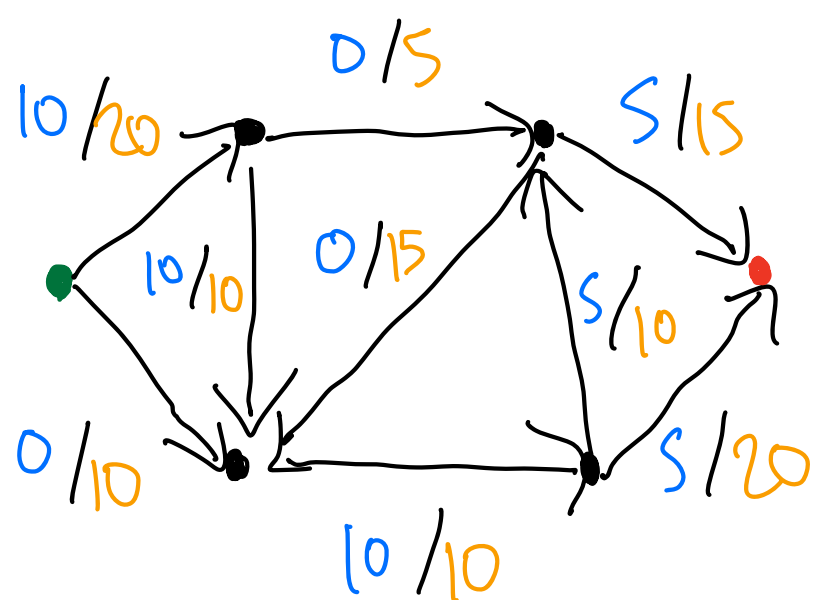
- If $f_{(u,v)} = 0$:

Only add forward edge

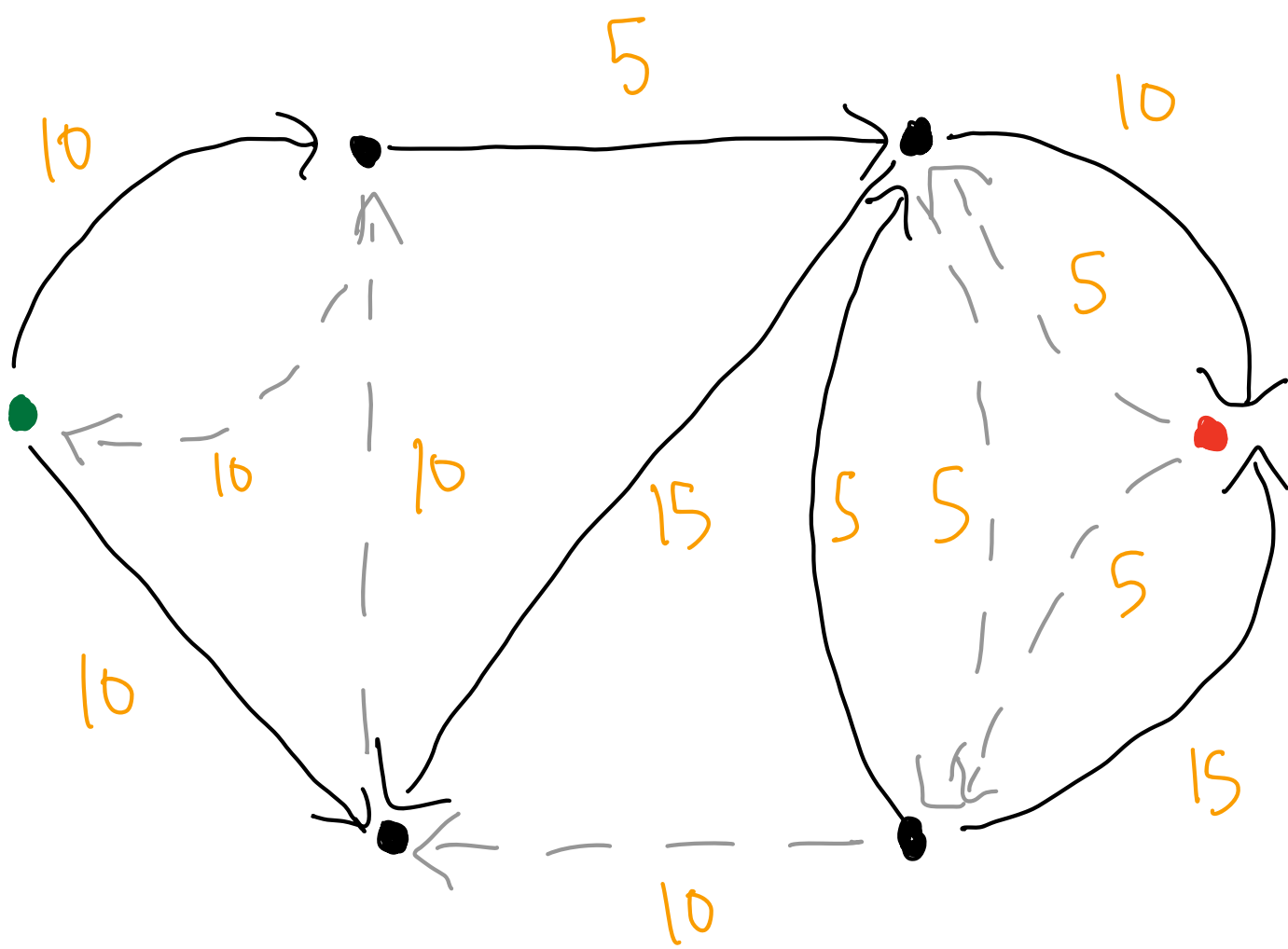
(u,v) to G^f , capacity = $f_{(u,v)}$

Examples

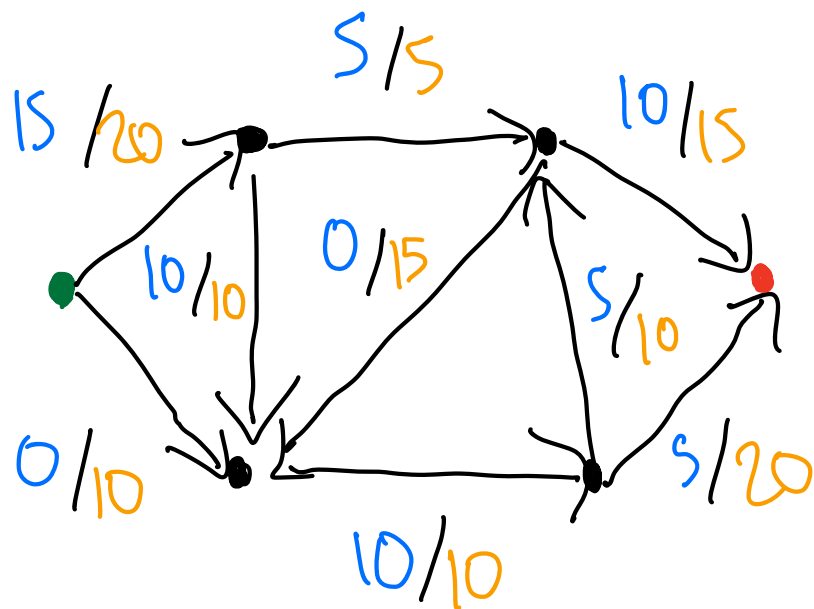
Flow on G :



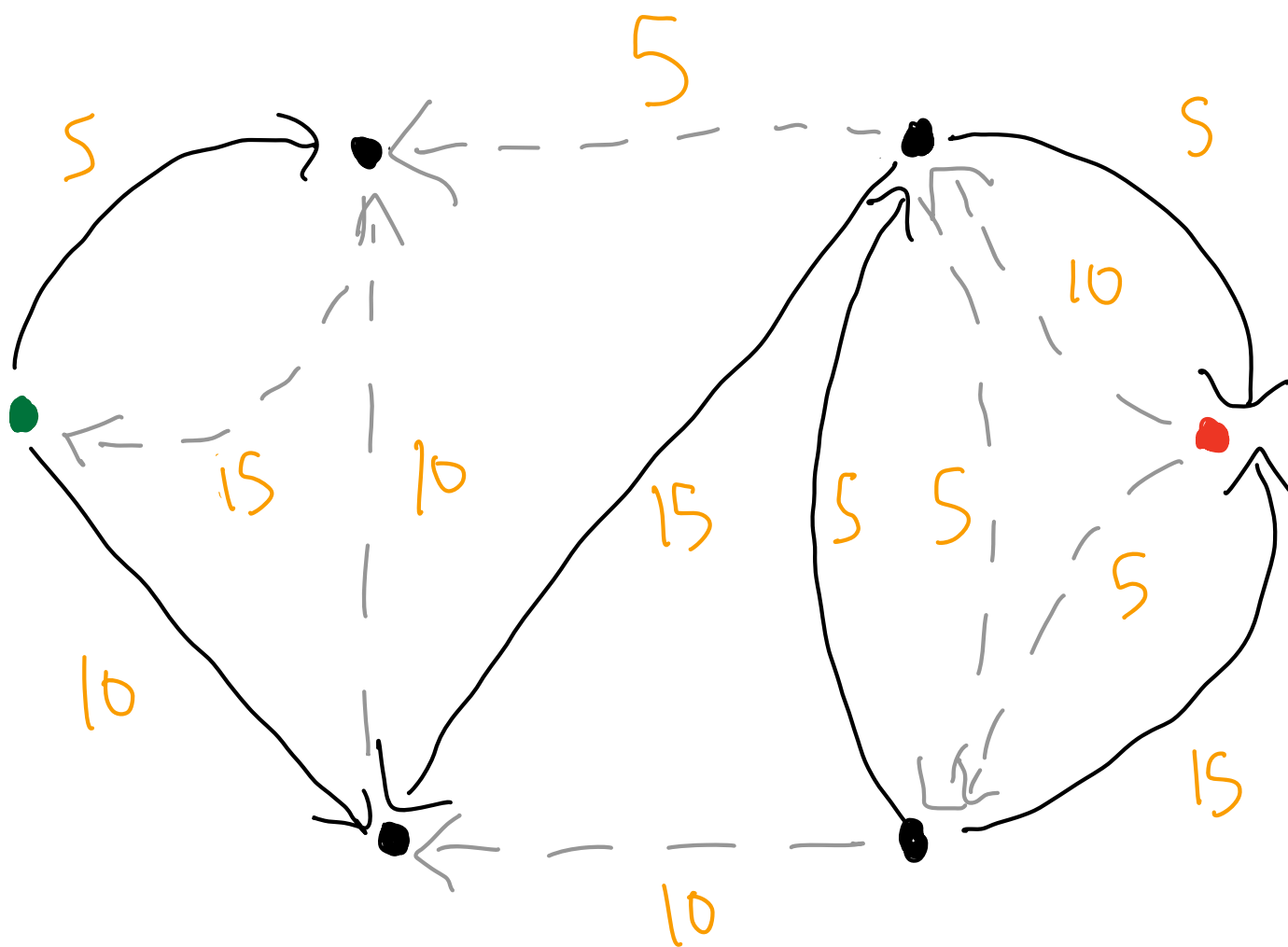
G^f :



Flow on G :



G^f :



Interestingly, s cannot reach t .

Proof of Strong maxflow-min cut:

Plan: 1) If $s-t$ path in G^f ,
not maxflow

2) If no $s-t$ path in G^f ,
 $f(s) = \text{cut}(S)$

where S reachable from s

Proof 1): Every edge in G^f has
positive capacity.

Let P be $s-t$ path,

$$w = \text{width}(P) > 0$$

We can create a bigger flow.

$$f'_e = \begin{cases} f_e + w & e \in P \\ f_e & e \notin P \end{cases} \quad (\text{push } w \text{ flow along } P)$$

Still feasible (by construction).

Still s - t flow:

$$\partial f'(v) = \partial f(v) - \underbrace{w + w}_{\text{if } v \text{ participates in } P} = 0$$

$v \notin \{s, t\}$

$$\begin{aligned} \partial f'(s) &= \partial f(s) + w \\ &> \partial f(s) \end{aligned}$$

Proof 2): Suppose s can't reach t .

Let $S = \text{reachable from } s$

$t \notin S$, no edges from $S \rightarrow V \setminus S$

Claim: f, S satisfy $\star$!!!

Let $u \in S, v \notin S$.

- If $(u, v) \in E$, $f_{(u,v)} = c_{(u,v)}$
or forward edge in $G^f \Rightarrow \Leftarrow$
- If $(v, u) \in E$, similarly $f_{(u,v)} = 0$

Hence $\partial f(S) = \text{cut}(S)$ $\square$

Flow algos (Part V, Section 4.3)

Generic algos so far:

- Search: explore unexplored vertex
- Shortest path: relax tense edge
- Flow: push w units along augmenting path
 $s \rightarrow \text{path} \in G^f$

$\text{Maxflow}(G, s, t)$:

$f \leftarrow$ all-zeros vector in $\mathbb{R}^E$

While $\exists P, s \rightarrow t$ path in G^f :

$w \leftarrow \text{width}(P)$

$f \leftarrow f + (w \text{ along } P)$

Return f

What path?

Idea 1: any path

Suppose all capacities integers.

Invariant: all capacities in G^f integers.

Thus can always push $w \geq 1$ flow/iter.

Runtime analysis:

Let F^* = maxflow value.

$$\underbrace{F^*}_{\# \text{ iters}} \times \underbrace{O(m)}_{\text{Cost of finding path, e.g. BFS}} = O(mF^*)$$

Is this "polynomial"?

Suppose Capacities $C \in [1, U]^E$
(upper bound)

$$F^* \leq mU \Rightarrow \text{runtime} = O(m^2 U)$$

But, input length = $O(m \log(U)) \dots$
"pseudopolynomial"

Idea 2: widest path

Claim: if maxflow value = F^*

Then widest path has $w \geq \frac{F^*}{m}$.

Proof sketch:

Key idea is flow decomposition into:

- "paths"
- "circulations" $(\partial f(v) = 0 \ \forall v)$

Aside: flow decomposition

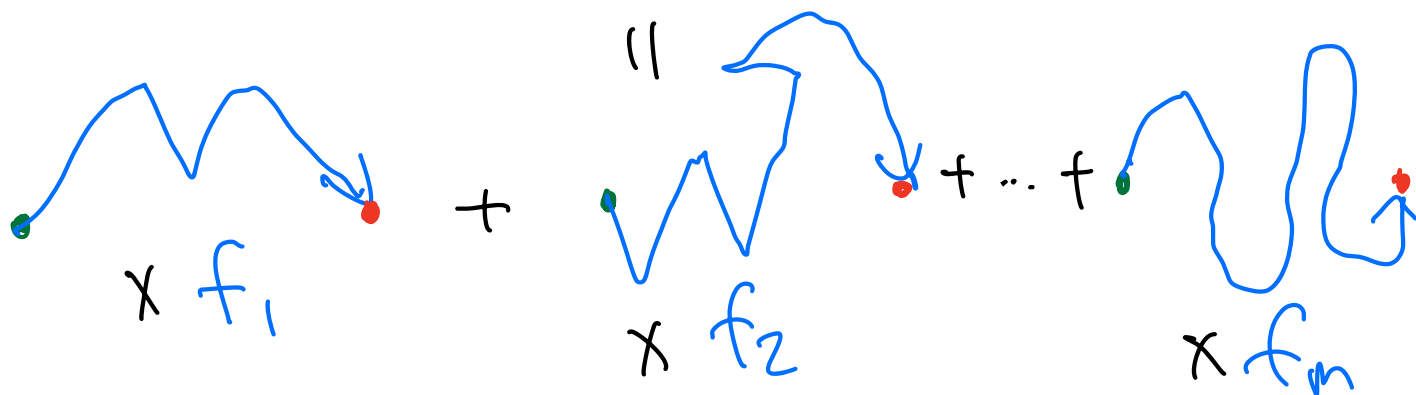
Every s - t flow can be decomposed into:

- $\leq m$ s - t paths $\times f_i \leftarrow$ flow value, path i
- one circulation ($\partial f(v) = 0$ for all v)

If $\partial f(s)$ is integer, then so are all f_i .



any s - t flow
value = $\partial f(s)$

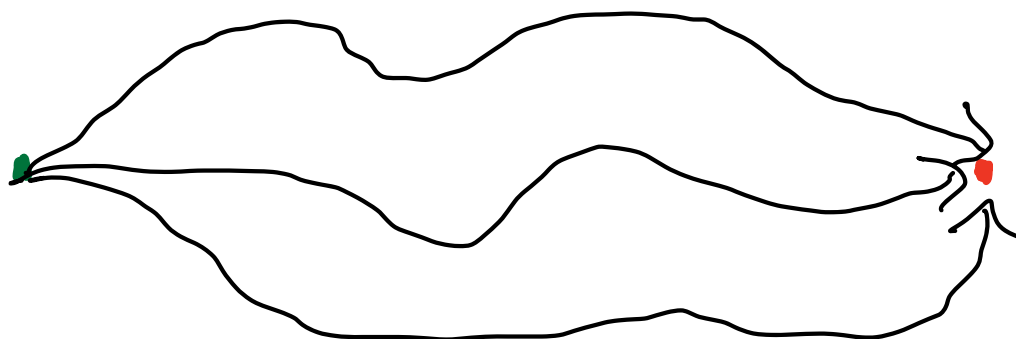


Very useful for flow reductions.

Thus, some path can carry $\frac{F^*}{m}$ flow

$\Rightarrow$ max width $\geq \frac{F^*}{m}$ as claimed.

total flow: F^*



$\leq m$ paths

Every iter decreases G^{maxflow} $(1 - \frac{1}{m})^x$

after $m \log(F^*)$ iters,

$$\text{maxflow} \leq \left(1 - \frac{1}{m}\right)^{m \log(F^*)} F^*$$

$$\leq \exp(-\log(F^*)) F^* \leq 1$$

(any path works)

Runtime: $O(m^2 \log(F^*))$ "weakly polynomial"

The frontier

Best strongly polynomial: $O(mn)$

(King- Rao- Tarjan '94, Orlin '13)

Best weakly polynomial: $O(m^{1+o(1)} \log(V))$

[Chen - King - Liu - Peng - Probst Gutenberg - Seidman '22]

Best pseudopolynomial: $\tilde{O}(m + \sqrt{mn}(F^*)^{3/4})$

(Sidford - Tian '18)

($\tilde{O} = \text{hide polylog}(n)$)